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Unbounded solution:

Example:

$$\begin{aligned} \max\ z &= 2x_1 + 3x_2 \\ x_1 - x_2 &\le 2 \\ \text{s.t}\quad -3x_1 + x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution: convert to standard form

$$\begin{aligned} \max\ z &= 2x_1 + 3x_2 + 0s_1 + 0s_2 \\ x_1 - x_2 + s_1 &= 2 \\ -3x_1 + x_2 + s_2 &= 4 \\ x_1, x_2, s_1, s_2 &\ge 0 \end{aligned}$$

We write the O.f $z - 2x_1 - 3x_2 - 0s_1 - 0s_2 = 0$

$= 0$

First bfs is z = 0, $x_1 = 0, x_2 = 0,\ s_1 = 2,\ s_2 = 4$

	NBV x_1	NBV x_2 ↓	BV s_1	BV s_2	b	
z	-2	-3	0	0	0	ratio
s_1	1	-1	1	0	2	2/-1
← s_2	-3	1	0	1	4	4/1
Z	-11	0	0	3	12	ratio
s_1	-2	0	1	1	6	------
x_2	-3	1	0	1	4	------

The LP is **unbounded**

When LP has all the constraints from type (≤) with nonnegative right-hand sides then the slack variables give basic feasible solution. Models involving (=) and/or (≥) constraints do not. So we need to use artificial variables that play the role of slacks at the first iteration. The artificial variables are then disposed of at a later iteration. Two closely related methods are introduced here: The M-method and the two-phase method.

Big-M Method

An artificial variable, a, is added to form a starting solution similar to the all-slack basic solution. However, because the artificial variables are not part of the original problem, a modeling "trick" is needed to force them to zero value by the time the optimum iteration is reached. The desired goal is achieved by assigning **a penalty** defined as:

Note: Artificial variable objective function coefficient -M in maximization problems, M in minimization problems (where M is large positive value).

What value of M should we use? The answer depends on the data of the original LP. Recall that the penalty M must be sufficiently large relative to the original objective coefficients to force the artificial variables to be zero (which happens only if a feasible solution exists).

Example:

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ \text{s.t} \quad x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution:

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 - s_1 &= 6 \\ x_1 + 2x_2 + s_2 &= 4 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

The third equation has its slack variable, s_2, but the first and second equations do not. When simplex starts with x1=x2=0, then s1= -6 and first constraint gives 0=3 which is not acceptable. Thus, we add the artificial variables a_1 and a_2 in the first

two equations and penalize them in the objective function with $Ma_1 + Ma_2$ (because we are minimizing).

$$\text{Minimize } z = 4x_1 + x_2 + 0s_1 + 0s_2 + Ma_1 + Ma_2$$

$$\begin{aligned} 3x_1 + x_2 \quad\quad + a_1 \quad\quad &= 3 \\ 4x_1 + 3x_2 - s_1 \quad\quad + a_2 &= 6 \\ x_1 + 2x_2 \quad\quad + s_2 &= 4 \\ x_1, x_2, s_1, s_2, a_1, a_2 &\geq 0 \end{aligned}$$

The starting basic solution is $(a_1, a_2, s_2) = (3,6,4)$

$$z - 4x_1 - x_2 - 0s_1 - 0s_2 - Ma_1 - Ma_2 = 0$$

Basic	x_1	x_2	s_1	s_2	a_1	a_2	r.h.s	
z	-4	-1	0	0	-M	-M	0	z-row0
a_1	3	1	0	0	1	0	3	a_1-row1
a_2	4	3	-1	0	0	1	6	a_2-row2
s_2	1	2	0	1	0	0	4	row3

نعيد كتابة الصف z بدلالة M

New z-row = Old z-row +(M* a_1-row + M* a_2-row)

-4	-1	0	0	-M	-M	0	old z-row +
3M	M	0	0	M	0	3M	a_1-row *M +
4M	3M	-M	0	0	M	6M	a_2-row *M
7M-4	4M-1	-M	0	0	0	9M	= New z-row

نضيف الصف الجديد z الى الجدول و نبدأ الحل باختيار اكبر عدد موجب ليكون متغير داخل ثم نحسب النسبة لاختيار المتغير الخارج

Because this problem is minimizing, we choose the variable with **most positive** coefficient to enter the basis.

Min ratio = {3/3, 6/4, 4/1} = {1, 1.5, 4} = 1

Basic	x_1	x_2	s_1	s_2	a_1	a_2	r.h.s	
z	-4	-1	0	0	-M	-M	0	
a_1	3	1	0	0	1	0	3	
a_2	4	3	-1	0	0	1	6	
s_2	1	2	0	1	0	0	4	
z	7M-4	4M-1	-M	0	0	0	9M	ratio
a_1	3	1	0	0	1	0	3	3/3=1
a_2	4	3	-1	0	0	1	6	6/4=1.5
s_2	1	2	0	1	0	0	4	4/1=4
z	0	$\frac{5M}{3}+\frac{1}{3}$	-M	0	$\frac{-7M}{3}+\frac{4}{3}$	0	4	
x_1	1	1/3	0	0	1/3	0	1	1/1/3=3
a_2	0	5/3	-1	0	-4/3	1	2	2/5/3=1.2
s_2	0	5/3	0	1	-1/3	0	3	3/5/3=1.8
z	0	0	1/5	0	$-M+2$	$-M-\frac{1}{5}$	$-2M+\frac{18}{5}$	
x_1	1	0	1/5	0	3/5	-1/5	3/5	3/5/1/5=3
x_2	0	1	-3/5	0	-4/5	3/5	6/5	
s_2	0	0	1	1	1	-1	1	1/1=1
z	0	0	0	-1/5	$-M+\frac{9}{5}$	$-M$	$-M+\frac{17}{5}$	
x_1	1	0	0	-1/5	2/5	0	2/5	
x_2	0	1	0	3/5	-1/5	0	9/5	
s_1	0	0	1	1	1	-1	1	

$$x_1 = \frac{2}{5}, \quad x_2 = \frac{9}{5}, \quad z = 4x_1 + x_2 = 4 * \frac{2}{5} + \frac{9}{5} = \frac{17}{5}$$

Note: that the artificial variables a1 and a2 leave the basic solution (i.e., become equal to zero) promptly in the first and second iterations, a result that is consistent with the concept of penalizing them in the objective function.

Calculations:

We choose x_1 as entering variable min ratio = {3/3, 6/4, 4/1} = 1 so a1 is leaving the basis.

New row1 = old row1 * 1/3

New row2 = old row2 – 4 * new row1

New row3 = old row3 – new row1

New z-row = old z-row – (7M-4) * new row1

Min ratio = {1/1/3, 2/5/3, 3/5/3}={3, 1.2, 1.8} = 1.2 so x_2 will enter and a2 will leave.

New row2 = 3/5 * old row2

New row1 = row1 -1/3 new row2

New row3 = row3 – 5/3 new row2

New z row = z-row - $(\frac{5M}{3}+\frac{1}{3})$ new row2

Min ratio = {3/5 / 1/5, 1/1}={3, 1}=1 so s1 entering the basis and s2 leaving.

New row3= old row3

New row1= row1 -1/5 new row3

New row2= row2 +3/5 new row3

New z-row = z-row -1/5 new row3

Note: you can use $M = 100$ in the table if you wish!!

H.W: solve the following LP

$$\begin{aligned} \min\ z = {} & 2x_1 + 3x_2 \\ & \frac{1}{2}x_1 - \frac{1}{4}x_2 \le 4 \\ \text{s.t} \quad & x_1 + 3x_2 \ge 20 \\ & x_1 + x_2 = 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max\ z = {} & 2x_1 + 3x_2 - 5x_3 \\ & x_1 + x_2 + x_3 = 7 \\ \text{s.t} \quad & 2x_1 - 5x_2 + x_3 \ge 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$